Normal Matrices*

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ABSTRACT

In hopes that it will be useful to a wide audience, a long list of conditions on an n-by-n complex matrix A, equivalent to its being normal, is presented. In most cases, a description of why the condition is equivalent to normality is given.

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I. INTRODUCTION

Our purpose is to present a list of conditions on an n-by-n complex matrix A, each of which is equivalent to A being normal. Presumably many of these are known, and no attempt at history or comprehensive literature survey is intended. Self-contained brief descriptions of proofs of most of the conditions are given in Section III; many elementary details (such as proof in an obvious direction) are omitted. References are given for the remaining conditions. Since, over the years, we have often noted that knowledge of even some of the simpler conditions would have been useful to various authors and since we know of no similar list, our hope is that this will be generally useful to the community.

For this purpose, we define A to be *normal* if and only if

$$A^*A = AA^*,$$

which may be taken to be condition 0 on the list appearing in the next section. Each of the 70 conditions listed in Section II is then equivalent to the normality of A, except that some involve indicated restrictions on A (e.g. nonsingularity or distinct eigenvalues). For brevity, definitions and notation found in most matrix-theory texts are omitted.

The point of view taken in compiling this list is that of matrices rather than operators. For example, in many of the conditions, the word "eigenvector" could be replaced by "invariant subspace," thereby making infinite-dimensional extensions apparent. Though many conditions we have listed are similar, the list could be expanded much further by including variations on the statement of commutativity, etc. Also, we have refrained from going beyond characterizations of the normality of a single matrix and not included results about sums or products of normal matrices, etc. However, a number of papers dealing with normal matrices and not explicitly referenced are included in Section IV. Some historical references are included, for example, in the old survey [4].

The condition of normality is a strong one, but, as it includes the Hermitian, unitary, and skew-Hermitian matrices, it is an important one which often appears as the appropriate level of generality in highly algebraic work and for numerical results dealing with perturbation analysis. Neither the list nor the self-contained selection of proofs is exhaustive, but, reflecting the fact that normality arises in many ways, it is hoped that not only will it be useful now, but its utility will grow over time as conditions are added.

II. CONDITIONS

1. $p(A)$ is normal for any polynomial.
2. A^{-1} is normal (for invertible A).
3. $A^{-1}A^*$ is unitary (for invertible A).
4. $A = A^*AA^{*-1}$ (for invertible A).
5. A commutes with $A^{-1}A^*$ (for invertible A).
6. $AB = BA$ implies $A^*B = BA^*$.
7. U^*AU is normal for any (or for some) unitary U.
8. For any unitary U for which

$$U^*AU = B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

with B_{11} square, the matrix $B_{12} = 0$.

9. If $W \subseteq C^n$ is an invariant subspace of A, then so is $W^{\perp}$.
10. If x is an eigenvector of A, then $x^{\perp}$ is an invariant subspace of A.
11. There exist unitary U and diagonal D such that $U^*AU = D$.
12. If x is an eigenvector of A, then x is an eigenvector A^*.
13. A has a full linearly independent set of eigenvectors, and any two eigenvectors corresponding to distinct eigenvalues are orthogonal.
14. There exists an orthonormal basis of $\mathbf{C}^n$ consisting of eigenvectors of A.
15. There exist complex scalars $\lambda_1, \ldots, \lambda_n$ such that the matrix A may be written as

$$A = \sum_{j=1}^{n} \lambda_j E_j, \qquad E_j \text{ } n\text{-by-}n,$$

in which

$$E_j^2 = E_j = E_j^*, \qquad E_j E_l = 0 \quad \text{for } j \neq l, \quad \text{and} \quad \sum_{j=1}^{n} E_j = I.$$

16. There exist distinct complex scalars $\lambda_1, \ldots, \lambda_k$ such that the matrix A may be written as

$$A = \sum_{j=1}^{k} \lambda_j P_j, \qquad P_j \text{ } n\text{-by-}n,$$

in which

$$P_j^2 = P_j = P_j^*, \qquad P_jP_l = 0 \quad \text{for } j \neq l, \quad \text{and} \quad \sum_{j=1}^{k} P_j = I.$$

17. There exists a polynomial p such that $A^* = p(A)$.
18. A commutes with some normal matrix with distinct eigenvalues.
19. A commutes with some Hermitian matrix with distinct eigenvalues.
20. $A^*A - AA^*$ is semidefinite.

[Henceforth, let $H = \frac{1}{2}(A + A^*)$, $K = \frac{1}{2}(A - A^*)$.]

21. $HK = KH$.
22. $AH = HA$.
23. $AH + HA^* = 2H^2$ $(= HA + A^*H)$.
24. $AK = KA$.
25. $AK - KA^* = 2K^2$ $(= KA - A^*K)$.
26. $H^{-1}A + A^*H^{-1} = 2I$ $(= AH^{-1} + H^{-1}A^*)$ (as long as A has no pure imaginary eigenvalues, or H is nonsingular).
27. $K^{-1}A - A^*K^{-1} = 2I$ $(= AK^{-1} - K^{-1}A^*)$ (as long as A has no real eigenvalues, or K is nonsingular).
28. Any eigenvector of H is an eigenvector of K (as long as H has distinct eigenvalues).
29. Any eigenvector of K is an eigenvector of H (as long as K has distinct eigenvalues).
30. Any eigenvector of H is an eigenvector of A (as long as H has distinct eigenvalues).
31. Any eigenvector of A is an eigenvector of H.
32. Any eigenvector of K is an eigenvector of A (as long as K has distinct eigenvalues).
33. Any eigenvector of A is an eigenvector of K.

[Henceforth, let $\delta(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ denote the spectrum of A, and let $\delta(H) = \{\alpha_1, \dots, \alpha_n\}$, $\delta(K) = \{i\beta_1, \dots, i\beta_n\}$.]

34. There exists a permutation $\sigma \in S_n$ such that

$$\delta(A) = \left\{\alpha_j + i\beta_{\sigma(j)} \middle| j = 1, \dots, n\right\}.$$

35. $\operatorname{Re}\delta(A) = \{\alpha_1, \dots, \alpha_n\}$.
36. $\operatorname{Im}\delta(A) = \{\beta_1, \dots, \beta_n\}$.

(Henceforth, assume $A = VP$ is a polar factorization of A into a unitary times a positive semidefinite matrix.)

37. $VP = PV$.
38. $AV = VA$.
39. $AP = PA$.
40. Any eigenvector of V is an eigenvector of P (as long as V has distinct eigenvalues).
41. Any eigenvector of P is an eigenvector of V (as long as P has distinct eigenvalues).
42. Any eigenvector of V is an eigenvector of A (as long as V has distinct eigenvalues).
43. Any eigenvector of A is an eigenvector of V (as long as A has distinct eigenvalues).
44. Any eigenvector of P is an eigenvector of A (as long as P has distinct eigenvalues).
45. Any eigenvector of A is an eigenvector of P (as long as A has distinct eigenvalues).

[Henceforth, let $\delta(V) = \{u_1, \dots, u_n\}$, $\delta(P) = \{\rho_1, \dots, \rho_n\}$.]

46. There exists a permutation $\sigma \in S_n$ such that

$$\delta(A) = \left\{u_j \rho_{\sigma(j)} \middle| j = 1, \dots, n\right\}.$$

47. $\text{modulus}(\delta(A)) = \{\rho_1, \dots, \rho_n\}$.
48. $\text{argument}(\delta(A)) = \{u_1, \dots, u_n\}$ (as long as A is nonsingular).
49. There exists a permutation $\sigma \in S_n$ such that

$$\delta(A^*A) = \left\{\lambda_j \bar{\lambda}_{\sigma(j)} \middle| j = 1, \dots, n\right\}.$$

50. There exists a permutation $\sigma \in S_n$ and nonzero complex α and β for which

$$\delta(\alpha A + \beta A^*) = \left\{\alpha \lambda_j + \beta \overline{\lambda_{\sigma(j)}} \middle| j = 1, \dots, n\right\}.$$

51. There exist complex $a_1, a_2, b_1, b_2, c_1, c_2$, with $c_1 + c_2 \neq 0$, which satisfy

$$\delta(a_1 A + a_2 A^* + b_1 A^2 + b_2 A^{*2} + c_1 A^* A + c_2 A A^*)$$
$$= \left\{a_1 \lambda_j + a_2 \overline{\lambda_j} + b_1 \lambda_j^2 + b_2 \overline{\lambda_j^2} + (c_1 + c_2) \lambda_j \overline{\lambda_j} \middle| j = 1, \dots, n\right\}$$

52. The same condition as 51, only that $a_2=\overline{a_1}$, $b_2=\overline{b_1}$, c_1, c_2 real, $c_1^2+c_2^2\neq 0$.

53. $|\lambda_1|^2+\cdots+|\lambda_n|^2=\operatorname{tr}(A^*A)=\Sigma|a_{ij}|^2$.

54. $(\operatorname{Re}\lambda_1)^2+\cdots+(\operatorname{Re}\lambda_n)^2=\alpha_1^2+\cdots+\alpha_n^2=\operatorname{tr}(H^2)$

55. $(\operatorname{Im}\lambda_1)^2+\cdots+(\operatorname{Im}\lambda_n)^2=\beta_1^2+\cdots+\beta_n^2=-\operatorname{tr}(K^2)$

56. If U is unitary and $\operatorname{diag}(U^*AU)=(\lambda_1,\dots,\lambda_n)$, then U^*AU is diagonal.

57. $\delta(A^*A)=\{|\lambda_1|^2,\dots,|\lambda_n|^2\}$.

58. The singular values of A are $\{|\lambda_1|,\dots,|\lambda_n|\}$.

59. If $s_1\geqslant\cdots\geqslant s_n\geqslant 0$ are the singular values of A, $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A, and $|\lambda_1|\geqslant\cdots\geqslant|\lambda_n|$, then

$$s_1\dots s_k=|\lambda_1\dots\lambda_k|,\qquad \text{all}\quad k=1,\dots,n.$$

For the next two conditions, let A^+ denote the Moore-Penrose generalized inverse.

60. A^+ is normal.

61. A^+A^* is a partial isometry. (B is a *partial isometry* iff B^*B is an orthogonal projection.)

62. $(Ax, Ay)=(A^*x, A^*y)$, all $x, y\in\mathbf{C}^n$.

63. $(Ax, Ax)=(A^*x, A^*x)$, all $x\in\mathbf{C}^n$.

64. $\|Ax\|=\|A^*x\|$, all $x\in\mathbf{C}^n$.

65. $A^*=UA$ for some unitary U.

For the next few characterizations we introduce the following terms. The *numerical range* or *field of values* of A is

$$F(A)=\{(Ax,x)\,|\,\|x\|=1\}.$$

$F(A)$ is a compact convex subset of $\mathbf{C}$ [9, 21, 12] which contains the spectrum of A.

The *kth numerical range* is

$$F_k(A)=\left\{\sum_{i=1}^{k}(Ax_i,x_i)\,\middle|\,x_1,\dots,x_k \text{ o.n.}\right\}.$$

$F_k(A)$ is a compact convex subset of C [1] which contains all possible sums of k eigenvalues of A.

The *numerical radius* of A is defined as

$$w(A) = \max_{\|x\|=1} |(Ax, x)|$$

$$= \max\{|\alpha| \mid \alpha \in F(A)\}.$$

66. A is unitarily similar to a direct sum of matrices, each having the property that the spectrum lies on the boundary of the numerical range.

67. If $\delta(A) = \{\lambda_1, \dots, \lambda_n\}$, $|\lambda_1| \geqslant \cdots \geqslant |\lambda_n|$, and $C_k(A)$ is the kth compound of A,

$$w(C_k(A)) = \prod_{i=1}^{k} |\lambda_i|, \qquad k = 1, \dots, n.$$

68. $F_k(A)$ is the convex hull of the points

$$\lambda_{i_1} + \cdots + \lambda_{i_k}, \qquad 1 \leqslant i_1 < \cdots < i_k \leqslant n,$$

for each $k = 1, \dots, n$.

69. The set $\{(x^*Ax, x^*A^*Ax) \mid x \in \mathbf{C}^n,\ x^*x = 1\}$ is a polyhedron.

70. The set $\{(2x^*Ax, x^*A^*Ax - x^*x \mid x \in \mathbf{C}^n,\ x^*A^*Ax + x^*x = 1\}$ is a polyhedron.

III. PROOFS AND REMARKS

Conditions 1–5, 7 can easily be seen to be equivalent to A being normal by resorting to the defining equation for normality and elementary properties of A^*.

Condition 9 is a geometric restatement of condition 8. Condition 9 implies condition 10. Applying condition 10 to the Schur triangular form of A yields condition 11, which implies that A is normal by using condition 7. If A is normal, then so is B by condition 7, and resorting to the defining equation for normality yields $B_{12} = 0$. Hence conditions 8, 9, 10, 11 are all equivalent to A being normal.

If A satisfies condition 6, then A can be seen to be normal by letting $B = A$. If A is normal and $AB = BA$, then by conditions 7 and 11 we can assume without loss of generality that A is diagonal, and then $A^*B = BA^*$ becomes apparent. Hence conditions 1–11 are equivalent. See also [7], [16], and [3] for facts related to condition 6 about normality of products of matrices.

Condition 12 can be established by letting U be a unitary matrix with first column equal to $x/\|x\|$, replacing A by U^*AU, and then appealing to condition 8. The fact that $B_{12}=0$ translates to x being an eigenvector of A^*, and so conditions 1–12 are equivalent.

Conditions 14 is a restatement of condition 11, and condition 14 is equivalent to condition 13. Hence conditions 1–14 are equivalent.

If A is normal with eigenvalues $\lambda_1,\dots,\lambda_n$ and corresponding orthonormal basis of eigenvalues $\{x_1,\dots,x_n\}$, then the matrices E_i in condition 15 can be set equal to $x_i x_i^*$, $i=1,\dots,n$. Conversely, any matrices $E_1,\dots,E_n$ satisfying $E_j^2=E_j^*$, $E_jE_l=0$ for $j\neq l$, and $E_1+\cdots+E_n=I$ must arise in this way from an orthonormal basis.

In condition 16, the matrices $P_1,\dots,P_k$ are simply the orthogonal projection onto the eigenspaces of A.

Condition 11 implies condition 17, since we may assume $A=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ and choose a polynomial of degree at most $n-1$ which satisfies $p(\lambda_i)=\overline{\lambda}_i$, $i=1,\dots,n$. Then $p(A)=A^*$. Since condition 17 trivially implies $A^*A=AA^*$, we now have that conditions 1–17 are equivalent.

Condition 11 makes conditions 18 and 19 clear, since we can assume A is diagonal. Conversely, if A commutes with some normal matrix with distinct eigenvalues, then $A=p(N)$ for some polynomial p, and hence A is normal by condition 1.

Condition 20 implies that A is normal, since A^*A and AA^* are positive semidefinite with equal trace, but the zero matrix is the only semidefinite matrix with trace 0. Hence, $A^*A-AA^*=0$ and A is normal.

Applying elementary matrix algebra and properties of A^* shows that each of conditions 21–25 are equivalent to A being normal. Conditions 26 and 27 are respectively equivalent to 23 and 25 for the cases H nonsingular, K nonsingular.

If H has distinct eigenvalues, then K commutes with H iff any eigenvector of H is an eigenvector of K. Hence condition 28 is a restatement condition 21 if H has distinct eigenvalues. The arguments showing that conditions 29–33 are each equivalent to A being normal are similar. Without the parenthetical restriction, condition 28, for example, can be seen to be stronger than normality by letting $H=I$ and K be a nonscalar matrix. This is indicative of the necessity of the parenthetical restrictions among conditions 28–48.

Normality implies condition 34, since we may assume without loss of generality that A,H,K are all diagonal by condition 11. Condition 34 implies each of conditions 35 and 36, and we will use condition 35 to illustrate the converses and hence establish the equivalence of conditions

1–36. Assume that $\mathrm{Re}[\delta(H+K)]=\delta(H)$. By replacing A with $A+\lambda I$ if necessary, we may assume that the distinct eigenvalues of H are $\alpha_1>\cdots>\alpha_k>0$ and that the distinct eigenvalues of K are $i\beta_1,\dots,i\beta_l$, where $\beta_1>\cdots>\beta_l>0$. We may also assume without loss of generality that

$$H=\begin{pmatrix}\alpha_1 I & 0\\ 0 & D\end{pmatrix},\qquad K=\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}$$

where D is diagonal with maximum eigenvalue λ_2. If x is any eigenvector of $A=H+K$ corresponding to any eigenvalue with real part α_1, then x must be of the form

$$\begin{pmatrix}y\\ 0\end{pmatrix}.$$

But then, a dimension argument establishes that the set of all such y must constitute the sum of the eigenspaces of A corresponding to eigenvalues with real part α_1. Since this subspace is invariant under A and H, it must be invariant under K, and so $K_{12}=0$. We next argue that we may assume that K_{11} is diagonal by condition 11, and finish by applying an induction on n to the matrix $D+K_{22}$.

If A is normal, then $A^*A=AA^*$ implies that V commutes with P^2, which then implies that V commutes with P, since P is positive semidefinite. If $VP=PV$, then A is normal, since a product of commuting normal matrices is normal. This establishes the equivalence of condition 37, which is in turn clearly equivalent to each of conditions 38 and 39.

If V has distinct eigenvalues, then P commutes with V iff any eigenvector of V is an eigenvector of P. Hence condition 40 is equivalent to condition 37 in this case. Similar arguments show the equivalence of conditions 41–45. Actually, in conditions 43 and 45 it need only be assumed that the geometric and algebraic multiplicity of the eigenvalue 0 of A are the same.

Conditions 46–48 can be seen to be equivalent to A being normal in much the same way as conditions 34–36.

Conditions 49–52 are implied by the normality of A, since we may assume that A is diagonal. The converses appear in [10], and we will show that condition 49 insures the normality of A.

Suppose $\delta(A^*A)=\{\lambda_j\overline{\lambda_{\sigma(j)}}\mid j=1,\dots,n\}$, and that $|\lambda_1|\geqslant\cdots\geqslant|\lambda_n|\geqslant 0$. Then the maximum eigenvalue of A^*A is at most $|\lambda_1|^2$. Assume A is lower triangular with main diagonal $(\lambda_1,\dots,\lambda_n)$. If $a_{i1}\neq 0$ for some $i\geqslant 2$, then the $(1,1)$ entry of A^*A is greater than $|\lambda_1|^2$, which implies that the maximum eigenvalue of A^*A is greater than $|\lambda_1|^2$ by Cauchy's interlacing theorem.

Hence $a_{21} = \cdots = a_{n1} = 0$ and $\lambda_1 = \lambda_{\sigma(1)}$. An induction on n yields that A is diagonal as well as $\lambda_i = \lambda_{\sigma(i)}$, $i = 1, \dots, n$.

Condition 53 can be seen to be equivalent to A being normal by considering A to be in Schur triangular form. From this we see that $|\lambda_1|^2 + \cdots + |\lambda_n|^2 \leqslant \operatorname{tr}(A^*A)$, with equality iff the Schur form is diagonal, which is equivalent to A being normal.

Conditions 54 and 55 are each implied by the normality of A, since we may assume that A, H, K are all diagonal. We can show that condition 54 implies that A is normal by assuming that A is in Schur triangular form with the main diagonal of A equal $(\lambda_1, \dots, \lambda_n)$. But then the main diagonal of H is $(\operatorname{Re}\lambda_1, \dots, \operatorname{Re}\lambda_n)$, from which we see that

$$(\operatorname{Re}\lambda_1)^2 + \cdots + (\operatorname{Re}\lambda_n)^2 \leqslant \operatorname{tr}(H^2) = \operatorname{tr}(H^*H),$$

with equality iff H (and hence A) is diagonal. The argument that condition 55 implies normality is the same, or one can interchange the roles of H and K by replacing A with iA.

The argument that condition 56 is equivalent to A being normal is essentially the same as the arguments for conditions 53–55.

Condition 57 implies condition 49, which implies that A is normal. The converse is clear, since we can assume that A is diagonal. Condition 58 is a restatement of condition 57. Condition 59 is also equivalent to condition 57. In fact condition 59 is the case of equality in the inequalities of H. Weyl.

To establish condition 60, suppose that $A = UDV$, where U, V are unitary and D is a nonnegative diagonal matrix. Then $A^+ = V^*D^+U^*$, from which it is clear that $(A^+)^* = (A^*)^+$ and $(A^+)^+ = A$. These elementary algebraic properties of A^+ make it clear that condition 60 is equivalent to A being normal.

To show that condition 61 is equivalent to normality amounts to showing that A is normal iff A has a singular-value decomposition $A = UDV$ where UV is diagonal. This is so by condition 58, for example. It is worth noting that A^+ can be replaced by the Drazin inverse in conditions 60 and 61.

Let $B = A^*A - AA^*$. Condition 62 holds iff $(Bx, y) = 0$ for all x, y iff $B = 0$ iff A is normal. Condition 63 holds iff $(Bx, x) = 0$ for all x iff $B = 0$ iff A is normal. Condition 64 is a restatement of condition 63. Condition 65 implies condition 64, which implies A is normal. If A is normal, then writing $A = V^*DV$ where V is unitary and D is diagonal makes it clear there exists unitary U such that $A^* = UA$. Hence conditions 1–65 are equivalent.

If A is normal, then A is unitarily diagonalizable, and so condition 66 holds. The converse is in [14] and follows from the fact that if λ is an eigenvalue on the boundary of $F(A)$, then the orthogonal complement of the corresponding eigenspace is an invariant subspace of A.

If A is normal, we may assume that A is diagonal, and hence that $C_k(A)$ is also diagonal. But then, it is clear that

$$w(C_k(A)) = |\lambda_1 \cdots \lambda_k|, \qquad \text{all} \quad k = 1, \ldots, n.$$

For the converse we begin by noting that

$$w(A) = |\lambda_1|$$

implies that every eigenvalue of A of modulus $|\lambda_1|$ corresponds to an eigenspace whose orthogonal complement is also an invariant subspace. Hence we may assume that A has the form

$$\begin{pmatrix} D & 0 \\ 0 & A_1 \end{pmatrix},$$

where D is m-by-m diagonal with eigenvalues of modulus $|\lambda_1|$ and A_1 is upper triangular with eigenvalues of modulus less than $|\lambda_1|$ in descending order of modulus on the main diagonal. Now we use the assumption that

$$w(C_{m+1}(A)) = |\lambda_1 \cdots \lambda_{m+1}|,$$

and consider $C_{m+1}(A)$. Since A is upper triangular, we have that $C_{m+1}(A)$ is also. The first row of $C_{m+1}(A)$ has diagonal entry equal $\lambda_1 \cdots \lambda_{m+1}$. Furthermore, if any off-diagonal entry of A in row $m+1$ is nonzero, then $C_{m+1}(A)$ will have a nonzero off-diagonal entry in row 1. This implies that $\lambda_1 \cdots \lambda_{m+1}$ is not on the boundary of $F(C_{m+1}(A))$, which contradicts our assumption on $w(C_{m+1}(A))$. Hence all off-diagonal entries of A in row $m+1$ must equal zero. Proceeding inductively in this fashion, we conclude that A is diagonal, and hence normal.

Since $F_k(U^*AU) = F_k(A)$ for any unitary U, we can assume that A is diagonal if A is normal. For a diagonal matrix it is clear that condition 68 holds, and so normality implies condition 68. The converse is in [6].

The sets defined in conditions 69 and 70, each of which has been called the *shell* of the matrix (operator) A, have been defined in [2] and [24]. Their equivalence to normality has been proven there. Note that each is a more precise version of the numerical range F, and that $F(A)$ being polygonal is necessary but not sufficient for A to be normal [14].

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